

Volatility-of-Volatility Risk

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Abstract

We show that time-varying volatility of volatility is a significant risk factor which affects both the cross-section and the time-series of index and VIX option returns, above and beyond volatility risk itself. Volatility and volatility-of-volatility movements are identified in a model-free manner from the index and VIX option prices, and correspond to the VIX and VVIX indices in the data. The VIX and VVIX have separate dynamics, and are only weakly related in the data. Delta-hedged returns for index and VIX options are negative on average, and are more negative for strategies which are more exposed to volatility and volatility-of-volatility risks. In the time series, volatility and volatility of volatility significantly predict delta-hedged returns with a negative sign. The evidence in the data is consistent with a no-arbitrage model which features time-varying market volatility and volatility-of-volatility factors which are priced by investors. In particular, volatility and volatility of volatility have negative market prices of risk, so that investors dislike increases in volatility and volatility of volatility.

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1 Introduction

Recent studies show that volatility risks significantly affect asset prices and the macroeconomy.¹ In the data, asset market volatility can be directly captured by the volatility index (VIX). Calculated from the cross-section of S&P500 option prices, the VIX index provides a risk-neutral forecast of the aggregate index volatility over the next 30 days. The VIX index exhibits substantial fluctuations, which in the data and in many economic models drive the movements in asset prices and risk premia. Interestingly, the volatility of the VIX index itself varies over time. Computed from VIX options in an analogous way to the VIX, the volatility-of-volatility index (VVIX) directly measures the risk-neutral expectations of the volatility of volatility in the financial markets. In the data, we find that the VVIX has separate dynamics from the VIX, so that fluctuations in volatility of volatility are not directly tied to movements in market volatility. The volatility-of-volatility risks are a significant risk factor which affects the time-series and the cross-section of index and VIX option returns, above and beyond volatility risks. The evidence in the data is consistent with a no-arbitrage model which features time-varying market volatility and volatility-of-volatility factors which are priced by the investors. In particular, volatility and volatility of volatility have negative market prices of risk, so that investors dislike increases in volatility and volatility of volatility, and demand a risk compensation for the exposure to these risks.

Our no-arbitrage model follows and extends the one-factor stochastic volatility specification of equity returns in Bakshi and Kapadia (2003). Specifically, we introduce a separate time-varying volatility-of-volatility risk factor which drives the conditional variance of the variance of market returns.² Both factors are priced in our model. We use the model to characterize the payoffs to delta-hedged equity and volatility options. The zero-cost, delta-hedged positions represent the gains on a long position in the option, continuously hedged by an offsetting short position in the underlying asset. As argued in Bakshi and Kapadia (2003), delta-hedged option payoffs are very useful to study volatility-related risks as they most cleanly isolate the exposures to volatility risks.³ Indeed, under a standard linear risk premium assumption, we show that the expected payoff on the delta-hedged position in equity index options consists of the risk compensations for both volatility and volatility-of-volatility risks. For volatility options, the expected payoffs only involve the compensation for volatility-of-volatility risks. The risk compensations are given by the product of the

¹See e.g. Bansal and Yaron (2004), Bloom (2009), Bansal, Kiku, Shaliastovich, and Yaron (2013), Fernandez-Villaverde and Rubio-Ramírez (2013) for the discussion of macroeconomic volatility risks, and Coval and Shumway (2001), Bakshi and Kapadia (2003), Campbell, Giglio, Polk, and Turley (2012) for market volatility risks.

²We use the terms “variance risk” and “volatility risk” interchangeably unless otherwise specified.

³For example, unlike delta-hedged positions, zero-beta straddles analyzed in Coval and Shumway (2001) are not dynamically rebalanced and may contain a significant time-decay option premium component.

market price of risk, the risk exposure of the asset, and the time-varying quantity of each source of risk. The model thus delivers clear, testable predictions for the expected option returns and their relation to volatility and volatility-of-volatility risks. In the model, if investors dislike volatility and volatility of volatility so that the market prices of these risks are negative, delta-hedged equity and VIX option gains are negative on average. In the cross-section, the average returns are more negative for option strategies which have higher exposure to the volatility and volatility-of-volatility risks. Finally, in the time series, higher volatility and volatility of volatility predict more negative delta-hedged option gains in the future.

To evaluate model implications for volatility and volatility risks, we use monthly observations on the implied and realized variances for the market index and the VIX, and index and VIX option price data over the 2006 - 2013 sample. We verify that the option-implied volatility measures capture meaningful economic information about the uncertainty in future market returns and market volatility in the data. Using predictive regressions, we show that the VIX is a significant predictor of the future realized variance of market returns, while the VVIX significantly forecasts future realized variation in the VIX index itself. Including both volatility measures at the same time, we find that the predictive power is concentrated with the corresponding factor (i.e., the VIX for market return volatility, the VVIX for VIX volatility), and the other variable has an insignificant impact. This evidence confirms that the measured VIX and VVIX indices can indeed separately capture volatility and volatility-of-volatility movements in the asset markets.

In the time-series, the VVIX behaves quite differently from the VIX, consistent with a setup of our model which separates market volatility from volatility of volatility. The VVIX is much more volatile, and is less persistent than the VIX. The correlation between the two series is 0.30. While both volatility measures share several common peaks, most notably during the financial crisis, other times of economic distress and economic uncertainty, such as the Eurozone debt crisis and flash crash in May 2010 and the U.S. debt ceiling crisis in August 2011, are characterized by large increases in the VVIX with relatively little action in the VIX. On average, the risk-neutral volatilities of the market return and market volatility captured by the VIX and VVIX exceed the realized volatilities of returns and the VIX. The difference between the risk-neutral and physical volatilities of market returns is known as the variance premium (variance-of-variance premium for the VIX), and the findings of positive variance and variance-of-variance premium suggest that investors dislike variance and variance-of-variance risks, and demand a premium for being exposed to these risks.

We next turn to the asset-price evidence from the equity index and VIX option markets. In line with our model, we consider discrete-time counterparts to the continuously-

rebalanced delta-hedged gains; this approach is similar to Bakshi and Kapadia (2003). Consistent with the evidence in previous studies, the average delta-hedged returns on out-of-the-money equity index calls and puts are significantly negative in our sample. The novel evidence in our paper is that the average delta-hedged returns on VIX options are also negative and statistically significant at all strikes, except for out-of-the-money puts which are marginally significant. Estimates of the loss for call options range from -0.57% of the index value for in-the-money VIX calls to -1.41% for out-of-the-money calls. The negative average returns on index and VIX options directly suggest that the market prices of volatility and volatility-of-volatility risks are negative.

We then show that the cross-sectional spreads in average option returns are significantly related to the volatility and volatility-of-volatility risks. In lieu of calculating exact model betas, we compute proxies for the option exposures to the underlying risks using the Black and Scholes (1973) vega and volga. Vega represents an increase in the Black-Scholes value of the option as the implied volatility increases by 1%, and thus provides an estimate for the exposure of equity options to volatility risks, and of VIX options to volatility-of-volatility risks. Volga is the second partial derivative of the option price with respect to the volatility, which we use to measure the sensitivity of the index option price to the volatility-of-volatility risks. Vega and volga vary intuitively with the moneyness of the option in the cross-section, and help us proxy for the betas of the options to the underlying risks. Empirically, we document that average option returns are significantly and negatively related to our proxies for volatility and volatility-of-volatility risks. Hence, using the cross-section of equity index options and VIX options, we find strong evidence for a negative market price of volatility and volatility-of-volatility risks.

Finally, we consider a predictive role of our volatility measures for the future option returns. In the model, expected delta-hedged gains are time-varying and are driven by the volatility and volatility of volatility (by volatility-of-volatility for VIX options). In particular, as option betas are all positive, when the market prices of volatility-related risks are negative, both volatility measures should forecast future returns with a negative sign. This model prediction is consistent with the data. The VIX and VVIX significantly negatively predict future index option returns, and the VVIX is a significant negative predictor of option returns on the VIX. Hence, using the cross-sectional and time-series evidence from the option markets, we find strong support that both volatility and volatility-of-volatility risks are separate priced sources of in the option markets, and have negative market prices of risks.

Related Literature. Our paper is most closely related to Bakshi and Kapadia (2003) who consider the implications of volatility risk for equity index option markets. We extend their

approach to include volatility-of-volatility risk, and bring evidence from VIX options. To help us focus on the volatility-related risks, we consider dynamic delta-hedging strategies where a long position in option is dynamically hedged by taking an offsetting position in the underlying. Delta-hedged strategies are also used in Bertsimas, Kogan, and Lo (2000), Cao and Han (2013) and Frazzini and Pedersen (2012), and are a standard risk management technique of option traders in the financial industry (Hull (2011)). In an earlier study, Coval and Shumway (2001) considers the returns on zero-beta straddles to identify volatility risk sensitive assets. Zhang and Zhu (2006) and Lu and Zhu (2010) highlight the nature and importance of volatility risks by analyzing the pricing of VIX futures. Also notably our analysis suggests that variance dynamics are richer than that of the square-root process typically considered in the literature - these findings are consistent with the results of Christoffersen, Jacobs, and Mimouni (2010) and Branger, Kraftschik, and Völkert (2014).

In a structural approach, Bollerslev, Tauchen, and Zhou (2009) consider a version of the Bansal and Yaron (2004) long-run risks model which features recursive utility and fluctuations in the volatility and volatility of volatility of the aggregate consumption process. They show that in equilibrium, investors require compensation for the exposure to volatility and volatility-of-volatility risks. With preference for early resolution of uncertainty, the market prices of the two risks are negative. As a result, the variance risk premium is positive on average, and can predict future equity returns. Bollerslev et al. (2009) and Drechsler and Yaron (2011) show that the calibrated version of such a model can account for the key features of equity markets and the variance premium in the data. Our empirical results in the paper are consistent with the economic intuition in these models and complement the empirical evidence in these studies.

Finally, it is worth noting that in our paper we abstract from jumps in equity returns, and focus on diffusive volatilities as the main drivers of asset prices and risk premia. For robustness, we confirm that our predictability results are robust to controlling for jump risk measures such as the slope of the implied volatility curve, realized jump intensity (Barndorff-Nielsen and Shephard (2006) and Wright and Zhou (2009)), and risk-neutral skewness (Bakshi, Kapadia, and Madan (2003)). Hence, we argue that the VIX and VVIX have a significant impact on option returns even in the presence of stock market and volatility jumps; we leave a formal treatment of jumps for future research. Reduced-form models which highlight the role of jumps include Bates (2000), Pan (2002), and Duffie, Pan, and Singleton (2000), among others.

Our paper proceeds as follows. In Section 2 we discuss our model which links expected delta-hedged equity and volatility option gains to risk compensations for volatility and volatility-of-volatility risk. In Section 3, we describe the construction of both the model-free

implied variance measures and high-frequency realized variance measures, and summarize their dynamics in the time-series. We show that the implied variances have a strong ability to forecast future realized variance. Section 4 provides the empirical evidence from option prices by empirically implementing the delta-hedged option strategies in our model. Section 5 presents robustness tests for alternative measures of variance, as well as robustness of the results in the presence of jump risks. Section 6 concludes the paper.

2 Model

In this section we describe our model for stock returns, and both equity and volatility option prices. Our model is an extension of Bakshi and Kapadia (2003) and features time-varying market volatility and volatility-of-volatility factors. Both volatility risks are priced, and affect the level and time-variation of the expected asset payoffs.

2.1 Dynamics of Equity and Equity Option Prices

Under the physical measure (\mathbb{P}), the stock price S_t evolves according to:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu(S_t, V_t, \eta_t)dt + \sqrt{V_t}dW_t^1, \\ dV_t &= \theta(V_t)dt + \sqrt{\eta_t}dW_t^2, \\ d\eta_t &= \gamma(\eta_t)dt + \phi\sqrt{\eta_t}dW_t^3,\end{aligned}\tag{2.1}$$

where dW_t^i are the Brownian motions which drive stock returns, the stock return variance, and the variance of the variance, for $i = 1, 2, 3$, respectively. The Brownian components can be correlated: $dW_t^i dW_t^j = \rho_{i,j} dt$ for all $i \neq j$. V_t is the variance of instantaneous returns and η_t is the variance of innovations in V_t . Note that the drift of the variance V_t only depends on itself, and not on the returns S_t or the volatility of volatility η_t . Similarly, the drift of the volatility of volatility η_t is a function of the volatility of volatility η_t .

Under the risk-neutral measure (\mathbb{Q}), the stock price S_t follows a similar process, where the drifts are adjusted by the risk compensations for the corresponding risks:

$$\begin{aligned}\frac{dS_t}{S_t} &= r_f dt + \sqrt{V_t}d\tilde{W}_t^1, \\ dV_t &= (\theta(V_t) - \lambda_t^V)dt + \sqrt{\eta_t}d\tilde{W}_t^2, \\ d\eta_t &= (\gamma(\eta_t) - \lambda_t^\eta)dt + \phi\sqrt{\eta_t}d\tilde{W}_t^3.\end{aligned}\tag{2.2}$$

In this representation, the \tilde{W}_t^i represent Brownian motions under the risk-neutral \mathbb{Q} measure. λ_t^V captures the risk compensation for the variance risk, and λ_t^η reflects the compensation for the innovations in the the variance of variance. If investors dislike variance and variance-of-variance risks, the two risk compensations are negative. In this case, the variances have higher drifts under the risk-neutral measure than under the physical measure.

Let $C_t(K, \tau)$ denote the time t price of a call option on the stock with strike price K and time to maturity τ . Assume the risk-free rate r_f is constant. To simplify the presentation, we further abstract from dividends. While we focus our discussion on call options, the case of put options follows analogously. Given the specified dynamics of the stock price under the two probability measures, the option price is given by a twice-differentiable function C of the state variables: $C_t(K, \tau) = C(S_t, V_t, \eta_t, t)$. By Itô's Lemma,

$$dC_t = \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + \frac{\partial C}{\partial \eta} d\eta_t + b_t dt, \quad (2.3)$$

for a certain drift component b_t .

The discounted option price $e^{-r_f t} C_t$ is a martingale under \mathbb{Q} and thus has zero drift. We use Itô's Lemma again to obtain that:

$$\frac{\partial C}{\partial S} S_t r_f + \frac{\partial C}{\partial V} (\theta(V_t) - \lambda_t^V) + \frac{\partial C}{\partial \eta} (\gamma(\eta_t) - \lambda_t^\eta) + b_t - r_f C_t = 0. \quad (2.4)$$

This implies that:

$$b_t = r_f \left(C_t - \frac{\partial C}{\partial S} S_t \right) - \frac{\partial C}{\partial V} (\theta(V_t) - \lambda_t^V) - \frac{\partial C}{\partial \eta} (\gamma(\eta_t) - \lambda_t^\eta). \quad (2.5)$$

Let $\Pi_{t,t+\tau}$ stand for the delta-hedged option gain for call options:

$$\Pi_{t,t+\tau} \equiv C_{t+\tau} - C_t - \int_t^{t+\tau} \frac{\partial C}{\partial S} dS_u - \int_t^{t+\tau} r_f \left(C_u - \frac{\partial C}{\partial S} S_u \right) du. \quad (2.6)$$

The delta-hedged option gain represents the gain on a long position in the option, continuously hedged by an offsetting short position in the stock, with the net balance earning the risk-free rate.

Combining equations (2.5) and (2.3) together, we obtain that the delta-hedged option gain for call options is given by,

$$\begin{aligned}\Pi_{t,t+\tau} &\equiv C_{t+\tau} - C_t - \int_t^{t+\tau} \frac{\partial C}{\partial S} dS_u - \int_t^{t+\tau} r_f \left(C_u - \frac{\partial C}{\partial S} S_u \right) du \\ &= \int_t^{t+\tau} \lambda_u^V \frac{\partial C}{\partial V} du + \int_t^{t+\tau} \lambda_u^\eta \frac{\partial C}{\partial \eta} du + \int_t^{t+\tau} \frac{\partial C}{\partial V} \sqrt{\eta_u} dW_u^2 + \int_t^{t+\tau} \frac{\partial C}{\partial \eta} \phi \sqrt{\eta_u} dW_u^3.\end{aligned}\tag{2.7}$$

Since the expectation of Itô integrals is zero, the expected delta-hedged equity option gains are given by:

$$\mathbb{E}_t [\Pi_{t,t+\tau}] = \mathbb{E}_t \left[\int_t^{t+\tau} \lambda_u^V \frac{\partial C}{\partial V} du \right] + \mathbb{E}_t \left[\int_t^{t+\tau} \lambda_u^\eta \frac{\partial C}{\partial \eta} du \right].\tag{2.8}$$

The expected option gains depend on the risk compensation components for the volatility and volatility-of-volatility risks (λ_t^V and λ_t^η), and the option price exposures to these two sources of risks ($\frac{\partial C}{\partial V}$ and $\frac{\partial C}{\partial \eta}$). For tractability and consistency with the literature, we assume that the risk premium structure is linear:

$$\lambda_t^V = \lambda^V V_t, \quad \lambda_t^\eta = \lambda^\eta \eta_t,\tag{2.9}$$

where λ^V is the market price of the variance risk and λ^η is the market price of the variance-of-variance risk. We can further operationalize (2.8) by applying Itô-Taylor expansions (Milstein (1995)). This gives us a linear factor model structure (see details in the Appendix):

$$\frac{\mathbb{E}_t [\Pi_{t,t+\tau}]}{S_t} = \lambda^V \beta_t^V V_t + \lambda^\eta \beta_t^\eta \eta_t.\tag{2.10}$$

The sensitivities to the risk factors are given by:

$$\beta_t^V = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^V > 0, \quad \beta_t^\eta = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^\eta > 0,\tag{2.11}$$

where $\Phi_{t,n}^V$ and $\Phi_{t,n}^\eta$ are positive functions which depend on the moneyness of the option and $\frac{\partial C}{\partial V}$ and $\frac{\partial C}{\partial \eta}$, respectively. Hence, the expected payoff on the delta-hedged option position combines the risk compensations for the volatility and volatility-of-volatility risks. The two risk compensations are given by the product of the market price of risk, the exposure of the asset to the corresponding risk, and the quantity of risk. In particular, options are positive-beta assets to both volatility and volatility-of-volatility risks. Hence, if investors dislike

volatility and volatility-of-volatility risks so that their market prices of risks are negative, the expected option payoffs are negative as well.

2.2 Dynamics of VIX Option Prices

The squared VIX index is the annualized risk-neutral expectation of the quadratic variation of returns from time t to $t + \tau$, given by:

$$VIX_t^2 = \frac{1}{\tau} \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{t+\tau} V_s ds \right]. \quad (2.12)$$

Given our model assumptions, the VIX index is a function of the stock market variance: $VIX_t = VIX(V_t)$. For example, in a linear model where the variance drift $\theta(V_t)$ is linear in V_t , the squared VIX is a linear function of the stock market variance V_t .

Let F_t be the time t price of a VIX futures contract expiring at $t + \tau$. Under no-arbitrage and continuous mark-to-market, F_t is a martingale under the risk-neutral measure \mathbb{Q} :

$$F_t = \mathbb{E}_t^{\mathbb{Q}} [VIX_{t+\tau}] = \mathbb{E}_t^{\mathbb{Q}} [VIX(V_{t+\tau})]. \quad (2.13)$$

Under our model structure, the futures price F is a function of the market variance V_t and volatility of volatility η . Under economically plausible scenarios, the futures price is monotone in the two volatility processes.⁴ Knowing F_t and η_t is sufficient for V_t , so we can re-write the economic states $\begin{bmatrix} V_t & \eta_t \end{bmatrix}$ in terms of $\begin{bmatrix} F_t & \eta_t \end{bmatrix}$.

Let C_t^* be the time t price of a VIX call option, whose underlying is a VIX forward contract. The option price is given by a twice differentiable function of the state variables C^* , so that $C_t^*(K, \tau) = C^*(F_t, \eta_t, t)$. By Itô's Lemma:

$$dC_t^* = \frac{\partial C^*}{\partial F} dF_t + \frac{\partial C^*}{\partial \eta} d\eta_t + b_t^* dt, \quad (2.14)$$

for a drift component b_t^* .

Under the risk-neutral measure \mathbb{Q} , the discounted VIX option price process $e^{-r_f t} C_t^*$ is a martingale, so it must have zero drift:

$$\frac{\partial C^*}{\partial F} \mathcal{D}^{\mathbb{Q}}[F_t] + \frac{\partial C^*}{\partial \eta} (\gamma(\eta_t) - \lambda_t^\eta) - r_f C_t^* + b_t^* dt = 0. \quad (2.15)$$

⁴See also Zhang and Zhu (2006), Lu and Zhu (2010), and Branger et al. (2014) for VIX futures pricing models.

This implies that

$$\begin{aligned}
b_t^* &= r_f C_t^* + \frac{\partial C^*}{\partial \eta} \lambda_t^\eta - \frac{\partial C^*}{\partial \eta} \gamma(\eta_t) - \frac{\partial C^*}{\partial F} \mathcal{D}^\mathbb{Q}[F_t] \\
&= r_f C_t^* + \frac{\partial C^*}{\partial \eta} \lambda_t^\eta - \frac{\partial C^*}{\partial \eta} \gamma(\eta_t),
\end{aligned} \tag{2.16}$$

where the second line follows since F_t is a martingale under \mathbb{Q} .

Combining the above with Equation (2.14), we obtain the equation for the delta-hedged VIX option gain:

$$\begin{aligned}
\Pi_{t,t+\tau}^* &= C_{t+\tau}^* - C_t^* - \int_t^{t+\tau} \frac{\partial C^*}{\partial F} dF_s - \int_t^{t+\tau} r_f C_s^* ds \\
&= \int_t^{t+\tau} \frac{\partial C^*}{\partial \eta} \lambda_s^\eta ds + \int_t^{t+\tau} \frac{\partial C^*}{\partial \eta} \phi \sqrt{\eta_s} dW_s^3.
\end{aligned} \tag{2.17}$$

The delta-hedged VIX option gain in Equation (2.17) is the counterpart to the delta-hedged equity option gain in Equation (2.7). The difference comes from the fact that short stock position serving as the hedge in the case of equity options is funded at r_f while for VIX futures the hedging position is zero cost.

Taking expectations, we can derive a corresponding expected gain on delta-hedged VIX options. Specifically, under the assumption that the risk premia are linear, we can show that (see the Appendix for details):

$$\begin{aligned}
\frac{\mathbb{E}_t [\Pi_{t,t+\tau}^*]}{F_t} &= \frac{1}{F_t} \int_t^{t+\tau} \mathbb{E}_t \left[\frac{\partial C_s^*}{\partial \eta_s} \lambda_s^\eta \right] ds \\
&= \lambda^\eta \beta_t^* \eta_t.
\end{aligned} \tag{2.18}$$

The delta-hedged VIX option exposure to volatility-of-volatility risks is defined as $\beta_t^* = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^*$. It is a positive function, which depends on the moneyness of the option and option sensitivity $\frac{\partial C^*}{\partial \eta}$. Notably, while delta-hedged equity options are exposed to both the volatility V_t and volatility-of-volatility risks η_t (see equation (2.10)), delta-hedged VIX strategies are exposed only to the volatility-of-volatility risks. This helps us identify the relative importance of the two risks in the data.

3 Variance Measures

3.1 Construction of Variance Measures

The VIX index is a model-free, forward-looking measure of implied volatility in the U.S. stock market, published by the Chicago Board Options Exchange (CBOE). The square of the VIX index is defined as in Equation (2.12) where $\tau = \frac{30}{365}$. Carr and Madan (1998), Britten-Jones and Neuberger (2000) and Jiang and Tian (2005) show that the VIX can be computed from the prices of call and put options with the same maturity at different strike prices:

$$VIX_t^2 = \frac{2e^{r_f\tau}}{\tau} \left[\int_0^{S_t^*} \frac{1}{K^2} P_t(K) dK + \int_{S_t^*}^{\infty} \frac{1}{K^2} C_t(K) dK \right], \quad (3.1)$$

where K is the strike price, C_t and P_t are the put and call prices, S_t^* is the fair forward price of the S&P500 index, and r_f is the risk-free rate. The VIX index published by the CBOE is discretized, truncated, and interpolated across the two nearest maturities to achieve a constant 30-day maturity.⁵ Jiang and Tian (2005) show through simulation analysis that the approximations used in the VIX index calculation are quite accurate.

Since February 2006, options on the VIX have been trading on the CBOE, which give investors a way to trade the volatility of volatility. As of Q3 2012, the open interest in front-month VIX options was about 2.5 million contracts, which is similar to the open interest in front-month S&P500 index option contracts.

We calculate our measure of the implied volatility of volatility using the same method as the VIX, applied to VIX options instead of S&P500 options. The index, which has since been published by the CBOE as the “VVIX index” in 2012 and back-filled, is calculated as:

$$VVIX_t^2 = \frac{2e^{r_f\tau}}{\tau} \left[\int_0^{F_t} \frac{1}{K^2} P_t^*(K) dK + \int_{F_t}^{\infty} \frac{1}{K^2} C_t^*(K) dK \right], \quad (3.2)$$

where F_t is the VIX futures price, and C_t^* , P_t^* are the prices of call and put options on the VIX, respectively.⁶ The squared VVIX is calculated from a portfolio of out-of-the-money call and put options on VIX futures contracts. It captures the implied volatility of VIX

⁵More details on the exact implementation of the VIX can be found in the white paper available on the CBOE website: <http://www.cboe.com/micro/vix>

⁶The official index is back-filled until 2007. We apply the same methodology and construct the index for an additional year back to 2006. The correlation between our measure of the VVIX and the published index is over 99% in the post-2007 sample. Our empirical results remain essentially unchanged if we restrict our sample to only the post-2007 period.

futures returns over the next 30-days, and is a model-free, forward-looking measure of the implied volatility of volatility.

In addition to the implied volatilities, we can also compute the realized volatilities for the stock market and the VIX. The construction here follows Barndorff-Nielsen and Shephard (2004) using high-frequency, intraday data.⁷ Realized variance is defined as the sum of squared high-frequency log returns over the trading day:

$$RV_t = \sum_{j=1}^N r_{t,j}^2. \quad (3.3)$$

Barndorff-Nielsen and Shephard (2004) show that RV_t converges to the quadratic variation as $N \rightarrow \infty$. We follow the standard approach of considering 5 minute return intervals. A finer sampling frequency results in better asymptotic properties of the realized variance estimator, but also introduces more market microstructure noise such as the bid-ask bounce discussed in Heston, Korajczyk, and Sadka (2010). Liu, Patton, and Sheppard (2013) show that the 5 minute realized variance is very accurate, difficult to beat in practice, and is typically the ideal sampling choice in most applications combining accuracy and parsimony.

We estimate two realized variance measures, one for the S&P500 and one for the VIX. For the S&P500, we use the S&P500 futures contract and the resulting realized variance will be denoted RV^{SPX} . For the VIX we use the spot VIX index and denote the resulting realized variance of the VIX by RV^{VIX} , which is our measure of the physical volatility of volatility.

3.2 Variance Dynamics

All of our variables are at the monthly frequency. The implied variance measures are given by the index values at the end of the month, and the realized variance measures are calculated over the past month and annualized.

Table 1 presents summary statistics for the implied and realized variance measures. While the average level of the VIX is about 24%, the average level of the VVIX is much higher at about 87%, which captures the fact that VIX futures returns are much more volatile than market returns: volatility, itself, is very volatile. $VVIX^2$ is also more volatile and less persistent than VIX^2 , with an AR(1) coefficient of 0.423 compared to 0.805 for VIX^2 . The VVIX exhibits relatively low correlation with the VIX, with a correlation coefficient of about 0.30. The mean of realized variance for S&P500 futures returns is

⁷The data is obtained from <http://www.tickdata.com>.

0.031, which corresponds to an annualized volatility of 17.6%. S&P500 realized variance is persistent and quite strongly correlated to the VIX index (correlation coefficient of 0.88) and much more weakly correlated to the VVIX index (correlation coefficient about 0.32). The realized variance of VIX is strongly related to the VVIX index (correlation of 0.53), and to a lesser extent, the VIX index (correlation of 0.38).

Figure 1 shows the time-series of the VIX and VVIX from February 2006 to June 2013. There are some common prominent moves in both series, such as the a peak during the financial crises. Notably, however, the VVIX also peaks during other times of economic uncertainty such as the summer of 2007 (quant meltdown, beginning of the subprime crisis), May 2010 (Eurozone debt crisis, flash crash) and August 2011 (U.S. debt ceiling crisis). During these events, the VIX experienced upward movements, but of a magnitude far smaller than the spikes in the VVIX. The plot suggests that the VVIX captures important uncertainty-related risks in the aggregate market, distinct from the VIX itself.

In Figure 2 we present time-series plots for both S&P500 and VIX realized variances. As shown in Panel A, the realized and implied variances of the stock market follow a similar pattern, and S&P500 realized variance is nearly always below the implied variance. There is a large spike in both series around the financial crisis in October 2008, at which point realized variance exceeded implied variance. The difference between the mean of VIX^2 and RV^{SPX} is typically interpreted as a variance premium, which is the difference between end-of-month model-free, forward-looking implied variance calculated from S&P500 index options and the realized variance of S&P500 futures returns over the past month. Unconditionally, the average level of the VIX (24%) is greater than the average level of the S&P500 realized volatility (17.6%), so that the variance premium is positive, consistent with the evidence in Bollerslev et al. (2009) and Drechsler and Yaron (2011). This implies that under the risk-neutral measure, volatility has a higher mean than under the physical probability measure. In turn, this evidence suggests that the market price of the volatility risk is negative.

Panel B of the Figure shows the time series of the realized and implied volatility of the VIX index. Generally, the implied volatility tends to increase at times of pronounced spikes in the realized volatility. The implied volatility is also high during other times of economic distress and uncertainty, such as May 2010 (Eurozone debt crisis and flash crash), and August 2011 (U.S. debt ceiling crisis). The VVIX largely follows the same pattern. During normal times, the VVIX is above the VIX realized variance, although during times of extreme distress we see the realized variance of VIX can exceed the VVIX. The average level of the VVIX (87%) is greater than the average level of the VIX realized volatility (73.4%), so that the volatility-of-volatility premium is also positive.⁸ Similar to our discussion of the

⁸Song (2013) shows that the average level of his VVIX measure, computed using numerical integration rather than the model-free VIX construction, is lower than the average realized volatility of VIX. One of the

variance premium, this evidence suggests that investors dislike volatility-of-volatility risks, and the market price of volatility-of-volatility risks is negative.

In addition to unconditional moments, we can also analyze the conditional dependence of volatility and volatility of volatility. Specifically, we consider the predictability of future realized variances by the VIX and VVIX, in spirit of Canina and Figlewski (1993), Christensen and Prabhala (1998), and Jiang and Tian (2005) who use option implied volatilities to predict future realized volatilities. We follow Christensen and Prabhala (1998) and Jiang and Tian (2005) and conduct our predictability regressions of future realized variances using monthly, non-overlapping samples. We follow a standard approach in the literature and consider both univariate and multivariate encompassing regressions to assess the predictability of future realized variances by the VIX and VVIX.

In our main specification, the dependent variable is the realized variance (RV) over the next month, for both the S&P500 and the VIX. Univariate regressions test whether each implied volatility measure (the VIX or the VVIX) can forecast future realized variances; multivariate encompassing regressions compare the relative forecasting importance of the VIX and VVIX and whether one implied volatility measure subsumes the information content of the other. The univariate regressions are restricted versions of the corresponding multivariate encompassing regression, which are presented below:

$$RV_{t+1}^{SPX} = \beta_0 + \beta_1 VIX_t^2 + \beta_2 VVIX_t^2 + \beta_3 RV_t^{SPX} + \epsilon_{t+1}, \quad (3.4)$$

Similarly for the VIX, we have:

$$RV_{t+1}^{VIX} = \beta_0 + \beta_1 VIX_t^2 + \beta_2 VVIX_t^2 + \beta_3 RV_t^{VIX} + \epsilon_{t+1}. \quad (3.5)$$

Our benchmark results are presented for all variables calculated in annualized variance units.

The first regression in Panel A of Table 2 shows that the VIX can forecast future realized variance of S&P500 returns. This is consistent with the findings of Jiang and Tian (2005). The VVIX can also forecast future S&P500 realized variance somewhat, although the statistical significance is weaker than that of the VIX and the magnitude of the regression coefficient is several times smaller. In the encompassing regression, we see that the VIX dominates the VVIX in forecasting future S&P500 realized variance. A one standard deviation increase in VIX^2 is associated with a 0.6 standard deviation increase in

key differences between his and our computations is the frequency of returns used in the realized variance computations. Consistent with the literature, we rely on 5-minute returns to compute the realized variances, while Song (2013) uses daily returns.

the realized variance of S&P500 returns next month. The coefficient on the VIX does not change much when we include the VVIX, which is consistent with our model. Including lags of the realized variances themselves do not materially change the results.

Panel B of Table 2 shows our predictability results for VIX realized variance, which is our proxy for physical volatility of volatility. The VIX is positively related to future VIX realized variation, but is not a significant predictor. The t-statistic is 0.95, and the adjusted R^2 is below zero. In stark contrast, the VVIX is a significant predictor of future VIX realized variation. The regression coefficient for the VVIX is about 0.8 in a univariate regression, and is largely unchanged in the multivariate regression. A one standard deviation increase in the current value of $VVIX^2$ is associated with more than one-third standard deviation increase in next month realized variance of VIX.

The empirical evidence suggests that fluctuations in the volatility of volatility are not directly related to the level of the volatility itself. This is consistent with our two-volatility model specification in Section 2. In many reduced form and structural models, the volatility of volatility is directly linked to the level of the volatility. For example, Heston (1993) models volatility as following a Cox, Ingersoll, and Ross (1985) square-root process. In that case, the level of volatility itself should forecast future realized volatility of volatility. The evidence in the data does not support this assumption, and calls for a richer dynamics of the volatility process, with separate movements in the volatility of volatility.

4 Evidence from Options

In this section, we analyze the implications of equity and VIX option price dynamics for the pricing of volatility and volatility-of-volatility risks in the data. Our economic model suggests that the market prices of volatility and volatility-of-volatility risks determine the key properties of the cross-section and time-series of delta-hedged equity and VIX option gains. Specifically, if market prices of volatility and volatility risks are negative, the average delta-hedged equity and VIX option gains are also negative. In the cross-section, the average returns are more negative to the option strategies which have higher exposure to the volatility and volatility-of-volatility risks. Finally, in the time series higher volatility and volatility of volatility predicts more negative gains in the future. We evaluate these model predictions in the data, and find a strong support that both volatility and volatility-of-volatility risks are priced in the option markets, and have negative market prices of risks.

4.1 Delta-Hedged Option Gains

We consider discrete-time counterparts to the continuously-rebalanced delta-hedged gains in Equations (2.7) and (2.17):

$$\begin{aligned}
 \Pi_{t,t+\tau} &= \underbrace{C_{t+\tau} - C_t}_{\text{option gain/loss}} - \underbrace{\sum_{n=0}^{N-1} \Delta_{t_n} (S_{t_{n+1}} - S_{t_n})}_{\text{delta hedging gain/loss}} + \underbrace{\sum_{n=0}^{N-1} r_f (\Delta_{t_n} S_{t_n} - C_t) \frac{\tau}{N}}_{\text{risk-free rate}}, \\
 \Pi_{t,t+\tau}^* &= \underbrace{C_{t+\tau}^* - C_t^*}_{\text{option gain/loss}} - \underbrace{\sum_{n=0}^{N-1} \Delta_{t_n} (F_{t_{n+1}} - F_{t_n})}_{\text{delta hedging gain/loss}} - \underbrace{\sum_{n=0}^{N-1} r_f C_t^* \frac{\tau}{N}}_{\text{risk-free rate}}.
 \end{aligned} \tag{4.1}$$

Δ_{t_n} indicates option delta, e.g. $\Delta_{t_n} = \frac{\partial C_{t_n}}{\partial S_{t_n}}$, and N is the number of trading days in the month. This discrete delta-hedging scheme is also used in Bakshi and Kapadia (2003) and Bertsimas et al. (2000).

At the close of each option expiration, we look at the prices of all options with non-zero open interest and non-zero trading volume. We take a long position in the option, and hedge the Δ each day according to the Black-Scholes model and hedge the Δ risk, with the net investment earning the risk-free interest rate appropriately.⁹ To minimize the effect of recording errors, we discard options that have implied volatilities below the 1st percentile or above the 99th percentile. All options have exactly one calendar month to maturity; S&P500 options expire on the third Friday of every month, while VIX options expire on the Wednesday that is 30 days away from the third Friday of the following month.

Table 3 shows average index and VIX delta-hedged option gains in our sample. We separate options by call or put, and group each option into four bins by moneyness to obtain eight bins for both S&P500 and VIX options. The first column $\frac{\Pi}{S}$ gives the delta-hedged option gain scaled by the index level, and the second column $\frac{\Pi}{C}$ gives the delta-hedged option gain scaled by the option price, which can be interpreted more readily as a “return” in the traditional sense. Panel A of Table 3 show that the average out-of-the-money delta-hedged S&P500 call options have significantly negative returns. Likewise, delta-hedged put options on the S&P500 also have significantly negative returns at all levels of moneyness. This evidence is largely consistent with Bakshi and Kapadia (2003), who focus on call options in an earlier sample period. In the model, negative average returns on delta-hedged index

⁹This requires an estimate of the implied volatility of the option, which may require an option price. We use implied volatilities directly backed out from market prices of options whenever possible; if an option does not have a quoted price on any intermediate date, we fit a cubic polynomial to the implied volatility curve given by options with quoted prices, and back out the option’s implied volatility. This is similar to typical option position risk management done by professional traders.

calls imply that volatility and/or volatility-of-volatility risks have a negative market price of risk. S&P500 option gains display mild positive serial correlation, which we will account for in our time-series predictive regressions in the later sections.

Panel B of Table 3 shows the average returns for delta-hedged VIX options. The average delta-hedged VIX option returns are negative and statistically significant in all bins except for out-of-the-money puts, which are marginally significant. Call options lose more money as they become more out of the money, regardless of whether we are scaling by the index or by the option price. Estimates of the loss for call options ranges from -0.57% of the index value for in-the-money VIX calls to -1.41% of the index value for out-of-the-money VIX calls. When viewed as a percentage of the option price, at-the-money delta-hedged VIX calls return about -10% per month. The results for VIX put options are similar. In the model, negative average returns for delta-hedged VIX options imply that investors dislike volatility-of-volatility risks, and are systematically paying a premium hedge against increases in the volatility of volatility. This suggests that the price of the volatility-of-volatility risk is negative. VIX option gains have small negative serial correlation, which we also account for in our time-series predictive regressions in the later sections. For both the S&P500 and VIX, the delta-hedged option gains are quite volatile.

In the next section, we provide further direct evidence by controlling for the exposures of the delta-hedged option positions to the underlying risks.

4.2 Cross-Sectional Evidence

As shown in the previous section, the average delta-hedged option gains are negative for S&P500 and VIX options. Our model further implies (see Equations (2.10) and (2.18)) that options with higher sensitivity (higher $\beta_t^V, \beta_t^\eta, \beta_t^*$) to volatility and volatility-of-volatility risks should have more negative gains. To compute the estimates of option exposures to the underlying risks, we follow the approach of Bakshi and Kapadia (2003) which relies on using the Black and Scholes (1973) model to proxy for the true option betas.

Specifically, to compute the proxy for the option beta to volatility risk, we consider the vega of the option:

$$\frac{\partial C}{\partial \sigma} = S \sqrt{\frac{\tau}{2\pi}} e^{-\frac{d_1^2}{2}} \propto e^{-\frac{d_1^2}{2}}, \quad (4.2)$$

where $d_1 = \frac{1}{\sigma\sqrt{\tau}} \left[\log \frac{S}{K} + \left(r_f - q + \frac{\sigma^2}{2} \right) \tau \right]$, q is the dividend yield, and σ is the implied volatility of the option. This approach allows us to compute proxies for the exposures of equity options to volatility risks, and of VIX options to the volatility-of-volatility risks.

To illustrate the relation between the moneyness of the option and the vega-measured exposure of options to volatility risks, we show the option vega as a function of option moneyness in Figure 3. Vega represents an increase in the value of the option as implied volatility increases by 1%. Higher volatility translates into higher future profits from delta-hedging due to the convexity effect; hence both call and put options have strictly positive vegas. Further, as the curvature of option value is the highest for at-the-money options, at-the-money options have the highest vega in the cross-section, and thus the largest exposure to the volatility risks. An alternative way to proxy for the option sensitivity to volatility risks is to use "gamma" of the option which represents the second derivative of the option price to the underlying stock price: $\Gamma = \frac{\partial^2 C}{\partial S^2}$. As shown in Figure 3, the shape of the vega and gamma functions are almost identical, hence, the implied cross-sectional dispersion in volatility betas by moneyness are very similar as well.

To capture the sensitivity of option price to the volatility of volatility, we compute the Black and Scholes (1973) second partial derivative of the option price with respect to the volatility, which is known in "volga" for "volatility gamma". Volga is calculated as:

$$\frac{\partial^2 C}{\partial \sigma^2} = S \sqrt{\frac{\tau}{2\pi}} e^{-\frac{d_1^2}{2}} \left(\frac{d_1 d_2}{\sigma} \right) = \frac{\partial C}{\partial \sigma} \left(\frac{d_1 d_2}{\sigma} \right), \quad (4.3)$$

$d_2 = d_1 - \sigma\sqrt{\tau}$. Figure 4 shows the plot of volga as a function of the moneyness of the option. Volga is positive, and exhibits twin peaks with a valley around at-the-money. At-the-money options are essentially pure bets on volatility, and are approximately linear in volatility (see Stein (1989)). Therefore, the volga is the lowest for at-the-money options. Deep-out-of-the-money options and deep-in-the-money options do not have much sensitivity to volatility of volatility either, since for the former it is a pure directional bet, and for the latter the option value is almost entirely comprised of intrinsic value. Options that are somewhat away from at-the-money are most exposed to volatility-of-volatility risks.

Table 4 shows our cross-sectional evidence from the regressions of average option returns on our proxies of options' volatility and volatility-of-volatility betas. Panel A shows univariate and multivariate regressions of delta-hedged S&P500 option gains scaled by the index on the sensitivities of the options to volatility and volatility-of-volatility risks. Our encompassing regression for delta-hedged S&P500 options is:

$$GAINS_{t,t+\tau}^i = \frac{\Pi_{t,t+\tau}^i}{S_t} = \tilde{\lambda}_1 VEGA_t^i + \tilde{\lambda}_2 VOLGA_t^i + \gamma_t + \epsilon_{t,t+\tau}^i. \quad (4.4)$$

Since each date includes multiple options, as in Bakshi and Kapadia (2003) we allow for a date-specific component in $\Pi_{t,t+1}^i$ due to the option expirations.¹⁰ Conceptually, our approach is related to Fama and MacBeth (1973) regressions. Instead of estimating risk betas in the first stage, due to the non-linear structure of option returns, we measure the our exposures from economically motivated proxies for the risk sensitivities.¹¹

The results in Panel A show that both volatility and the volatility of volatility are priced in the cross-section of delta-hedged S&P500 option returns. Options more exposed to volatility and volatility-of-volatility risks have more negative expected returns. The univariate estimates for vega and volga are -0.051 and -0.007. Both t-statistics are highly significant at conventional levels. In the multivariate regression, we see that the statistical significance becomes stronger for both risks, and the point estimates become larger: -0.178 for vega and -0.019 for volga. The signs and significance of these coefficients implies a significant negative volatility-of-volatility risk premium.

Panel B of Table 4 presents cross-sectional results for delta-hedged VIX options. As Equation (14) shows, delta-hedged VIX options are no longer exposed to volatility risk, and the vega for VIX options captures the sensitivity of VIX options to innovations in the volatility of volatility. The coefficient on vega of -1.46 is negative and statistically significant. Thus, in the cross-section of both S&P500 options and VIX options, we find strong evidence of a negative price of volatility-of-volatility risk.

4.3 Time-Series Evidence

In the model, time-variation in the expected delta-hedged option gains is driven by V_t and η_t , and the loadings are determined by the market prices of volatility and volatility-of-volatility risk. We group options into the same bins as we used for average returns in Table 3, and average the scaled gains within each bin, so that we have a time-series of option returns for each moneyness bin. To examine the contribution of both risks for the time-variation in expected index option payoffs, we consider the following regression:

$$GAIN S_{t,t+\tau}^i = \frac{\Pi_{t,t+\tau}^i}{S_t} = \beta_0 + \beta_1 VIX_t^2 + \beta_2 VVIX_t^2 + \gamma GAIN S_{t-\tau}^i + u^i + \epsilon_{t+\tau}^i. \quad (4.5)$$

where we include fixed effects u^i to account for the heterogeneity in the sensitivity of options in different moneyness bins to the underlying risks. We regress the delta-hedged option gain

¹⁰Muravyev (2012) highlights options market anomalies around expirations and relates it to inventory management concerns of options dealers.

¹¹Song and Xiu (2013) demonstrate an alternative method of estimating risk sensitivities nonparametrically using local linear regression methods.

scaled by the index from expiration to expiration on the value of the VIX and VVIX indices at the end of the earlier expiration; in other words, we run one-month ahead predictive regressions of delta-hedged option returns on the VIX and VVIX. We include lagged gains to adjust for serial correlation in the residuals, following Bakshi and Kapadia (2003).

Panel A of Table 5 shows the regression results for the index options. The univariate regression of delta-hedged S&P500 option gains on VIX^2 is negative and statistically significant, which is consistent with Bakshi and Kapadia (2003). The second regression is a multivariate regression of delta-hedged option gains on $VVIX^2$, which shows that both the VIX and VVIX loadings are negative and statistically significant. A one standard deviation increase in VIX^2 is associated with a -0.047% (of the S&P500 index value) lower delta-hedged option gain. In the same regression, a one standard deviation increase in $VVIX^2$ is associated with a -0.10% (of the S&P500 index value) lower delta-hedged option gain. Hence, both volatility and volatility of volatility command negative prices of risk in the S&P500 options market, and have significant contribution to the fluctuations in expected option returns.

Panel B of Table 5 shows the corresponding evidence for VIX options. The VVIX negatively and significantly predicts future VIX option gains. This is consistent with a negative market price of volatility-of-volatility risk.

5 Robustness

5.1 Alternative Variance Specifications

Our results for the predictability of realized by implied variance are robust to alternative specifications of volatility. Specifically, we consider regressing in volatility units or log-volatility units, rather than variance. The robustness regressions follow the form:

$$\sigma_{t+1}^x = \beta_0 + \beta_1 VIX_t + \beta_2 VVIX_t + \epsilon_{t+1} \quad (5.1)$$

$$\ln \sigma_{t+1}^x = \beta_0 + \beta_1 \ln VIX_t + \beta_2 \ln VVIX_t + \epsilon_{t+1} \quad (5.2)$$

where x refers to SPX or VIX .

In Table 6, we see that the point estimates and significance are very close to our baseline specification in variance units. In fact, the log-volatility specifications have coefficients much closer to 1 for the S&P500 predictability results, suggesting that the VIX is generally an unbiased forecast of future realized S&P500 volatility over the next month.

5.2 Sensitivity to Jump Risk Measures

The evidence in our paper highlights the roles of the volatility and volatility-of-volatility factors, which are driven by smooth Brownian motion shocks. In principle, the losses on delta-hedged option portfolios can also be attributed to large, discontinuous movements (jumps) in the stock market and in the market volatility. In this Section we verify that our empirical evidence for the importance of the volatility-related factors is robust to the inclusion of jump measures considered in the literature.

Specifically, we consider three measures of jump risks, which we construct for the S&P 500 returns and the VIX. Our first jump measure corresponds to the slope of the implied volatility curve:

$$\begin{aligned} SLOPE^{SPX} &= \sigma_{OTM}^{SPX} - \sigma_{ATM}^{SPX}, \\ SLOPE^{VIX} &= \sigma_{OTM}^{VIX} - \sigma_{ATM}^{VIX}. \end{aligned} \tag{5.3}$$

The *OTM* contract for the S&P500 options is defined as a put option with a moneyness closest to 0.9, and for VIX options as a call option with a moneyness closest to 1.1. In both cases, the *ATM* option has moneyness of 1. These slopes are positive for both index and VIX options. Positive slope of the index volatility smile is consistent with the notion of negative jumps in market returns (see e.g. Bates (2000), Pan (2002), Eraker, Johannes, and Polson (2003)), while the fact that the implied volatility curve for VIX options slopes upwards (call options are more expensive than put options on average) is consistent with the positive volatility jumps (Drechsler and Yaron (2011) and Eraker and Shaliastovich (2008), among others). In this sense, these slope measures help capture the variation in the market and volatility jumps in the economy.

Our second jump measure incorporates the whole cross-section of option prices, beyond just the slope of the smile. It is based on the model-free risk-neutral skewness of Bakshi et al. (2003):

$$SKEW(t, t + \tau) = \frac{e^{rf\tau}W_{t,t+\tau} - 3\mu_{t,t+\tau}e^{rf\tau}V_{t,t+\tau} + 2\mu_{t,t+\tau}^3}{[e^{rf\tau}V_{t,t+\tau} - \mu_{t,t+\tau}^2]^{3/2}}, \tag{5.4}$$

where $V_{t,t+\tau}$, $W_{t,t+\tau}$, $X_{t,t+\tau}$ are given by the prices of the volatility, cubic, and quartic contracts. Importantly, these measures are computed model-free using the observed option prices. The details for the computations are provided in the Appendix.

Finally, our third measure of jump risks is based on the high-frequency index and VIX data, rather than the option prices. It corresponds to the realized jump intensity, and

relies on the bipower variation methods in Barndorff-Nielsen and Shephard (2004), Huang and Tauchen (2005), and Wright and Zhou (2009). Specifically, while the realized variance defined in (3.3) captures both the continuous and jump variation, the bipower variation, defined as:

$$BV_t = \frac{\pi}{2} \left(\frac{M}{M-1} \right) \sum_{j=2}^M |r_{t,j-1}| |r_{t,j}| \quad (5.5)$$

measures the amount of continuous variation returns. Hence, we can use the test statistic to determine if there is a jump on any given day:

$$J_t = \frac{\frac{RV_t - BV_t}{RV_t}}{\sqrt{\frac{\theta}{M} \max\left(1, \frac{QV_t}{BV_t^2}\right)}}, \quad (5.6)$$

where $\theta = \left(\frac{\pi}{2}\right)^2 + \pi - 5$, and QV_t is the quad-power quarticity defined in Huang and Tauchen (2005) and Barndorff-Nielsen and Shephard (2004). The test statistic is distributed as $\mathcal{N}(0, 1)$. We flag the day as having a jump if the probability exceeds 99.9% both for index returns and for the VIX. These cut-offs imply an average frequency of jumps of once every two months for the index, and about three jumps a month for the VIX. This is broadly consistent with the findings of Tauchen and Todorov (2011), who find that VIX jumps tend to happen much more frequently than S&P500 jumps. Over a month, we sum up all the days where we have a jump, and we define our jump intensity measure on a monthly level as:

$$RJ = \frac{1}{T} \sum_{i=0}^{T-1} J_{t+i},$$

where T is the number of trading days in the month.

We use the jump statistics to document the robustness of the link between the volatility and volatility-of-volatility factors and options gains. We consider a regression:

$$GAINS_{t,t+\tau}^i = \beta_0 + \beta_1 VIX_t^2 + \beta_2 VVIX_t^2 + \beta_3 JUMP_t + \gamma GAINS_{t-\tau}^i + u^i + \epsilon_{t+\tau}^i \quad (5.7)$$

where $JUMP_t$ is one of the above jump risk proxies. We use index jump measures for index gains, and VIX jump measures for VIX gains. Table 7 displays our results. Both for S&P500 and VIX options, controlling for *SLOPE* does not change the ability of *VIX* and *VVIX* to predict future delta-hedged option gains. Both factors are still significant, and the point estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ are largely unchanged. *SLOPE* itself is not significant

at conventional levels for S&P500 options or VIX options. When we control for realized jump intensity RJ , we see a similar result where the statistical significance of VIX and $VVIX$ as well as their point estimates are largely unchanged. Neither for S&P500 nor for VIX options, RJ does not seem to be a significant predictor of future delta-hedged option gains. For VIX options, RJ does not affect the point estimate on $VVIX$ nor its significance; however, RJ does seem to be a predictor of future VIX option gains. Finally, risk-neutral skewness also does not affect the predictive ability of VIX and $VVIX$. While the skewness measures as insignificant themselves, the estimates have the correct sign since skewness is negative for S&P500 options and positive for VIX options; this is broadly similar to the findings of Bakshi and Kapadia (2003).

Hence, our evidence suggests that the VIX and VVIX have a significant impact on option returns even in the presence of stock market and volatility jumps. We leave a formal treatment of jumps for future research.

6 Conclusion

Using S&P500 and VIX options data, we show that a time-varying volatility of volatility is a separate risk factor which affects the option returns, above and beyond volatility risks. We measure volatility risks using the VIX index, and volatility-of-volatility risk using the VVIX index. The two indices, constructed from the index and VIX option data, capture the ex-ante risk-neutral uncertainty of investors about future market returns and VIX innovations, respectively. The VIX and VVIX have separate dynamics, and are only weakly related in the data: the correlation between the two series is 0.30. On average, risk-neutral volatilities identified by the VIX and VVIX exceed the realized physical volatilities of the corresponding variables in the data. Hence, the variance premium and variance-of-variance premium for VIX are positive, which suggests that investors dislike variance and variance-of-variance risks.

We show the pricing implications of volatility and volatility-of-volatility risks using options market data. Average delta-hedged option gains are negative, which suggests that investors pay a premium to hedge against innovations in not only volatility but also the volatility of volatility. In the cross-section of both delta-hedged S&P500 options and VIX options, options with higher sensitivities to volatility-of-volatility risk earn more negative returns. In the time-series, higher values of the VVIX predict more negative delta-hedged option returns, for both S&P500 and VIX options.

Our findings are consistent with a no-arbitrage model which features time-varying market volatility and volatility-of-volatility factors. The volatility factors are priced by the investors, and in particular, volatility and volatility of volatility have negative market prices of risks.

A Delta-Hedged Equity Options

The state vector is $x_t = [S_t \ V_t \ \eta_t]'$. Under the linear risk premium structure, $\lambda_t^V = \lambda^V V_t$ and $\lambda_t^\eta = \lambda^\eta \eta_t$. Note that since C_t is homogeneous of degree 1 in the underlying S_t , $\frac{\partial C}{\partial V}$ and $\frac{\partial C}{\partial \eta}$ are also homogeneous of degree 1 in S_t . Define a pair of functions:

$$\begin{aligned} g_1(x_t) &= \lambda_t^V \frac{\partial C_t}{\partial V_t} = \lambda^V V_t h_t^1(\tau; y) S_t \\ g_2(x_t) &= \lambda_t^\eta \frac{\partial C_t}{\partial \eta_t} = \lambda^\eta \eta_t h_t^2(\tau; y) S_t. \end{aligned} \tag{A.1}$$

We can re-write Equation (2.8) as:

$$\mathbb{E}_t [\Pi_{t,t+\tau}] = \mathbb{E}_t \left[\int_t^{t+\tau} g_1(x_u) du \right] + \mathbb{E}_t \left[\int_t^{t+\tau} g_2(x_u) du \right]. \tag{A.2}$$

Define operators \mathcal{L} and Γ such that:

$$\begin{aligned} \mathcal{L}[\cdot] dt &= \frac{\partial[\cdot]}{\partial S} \mu_t S_t dt + \frac{\partial[\cdot]}{\partial V} \theta(V_t) dt + \frac{\partial[\cdot]}{\partial \eta} \gamma(\eta_t) dt + \frac{\partial[\cdot]}{\partial t} dt \\ &+ \frac{1}{2} \frac{\partial^2[\cdot]}{\partial S^2} [dS_t, dS_t] + \frac{1}{2} \frac{\partial^2[\cdot]}{\partial V^2} [dV_t, dV_t] + \frac{1}{2} \frac{\partial^2[\cdot]}{\partial \eta^2} [d\eta_t, d\eta_t] \\ &+ \frac{\partial^2[\cdot]}{\partial S \partial \eta} [dS_t, d\eta_t] + \frac{\partial^2[\cdot]}{\partial S \partial V} [dS_t, dV_t] + \frac{\partial^2[\cdot]}{\partial V \partial \eta} [dV_t, d\eta_t] \\ \Gamma[\cdot] &= \left[\frac{\partial[\cdot]}{\partial S} S_t \sqrt{V_t}, \frac{\partial[\cdot]}{\partial V} \sqrt{\eta_t}, \frac{\partial[\cdot]}{\partial \eta} \phi \sqrt{\eta_t} \right]. \end{aligned} \tag{A.3}$$

Then, for $u > t$, Itô's Lemma implies that:

$$g_1(x_u) = g_1(x_t) + \int_t^u \mathcal{L}g(x_{u'}) du' + \int_t^u \Gamma g(x_{u'}) dW_{u'}.$$

The integral in the first expectation on the right-hand side of Equation (A.2) becomes:

$$\begin{aligned} \int_t^{t+\tau} g_1(x_u) du &= \int_t^{t+\tau} \left[g_1(x_t) + \int_t^u \mathcal{L}g(x_{u'}) du' + \int_t^u \Gamma g(x_{u'}) dW_{u'} \right] du \\ &= g_1(x_t) \tau + \frac{1}{2} \mathcal{L}g_1(x_t) \tau^2 + \frac{1}{6} \mathcal{L}^2 g_1(x_t) \tau^3 + \dots + \text{Itô Integrals} \\ &= \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \mathcal{L}^n g_1(x_t) + \text{Itô Integrals}, \end{aligned}$$

and likewise for the second integral in (A.2). We can use this to re-write (A.2) as:

$$\begin{aligned} \mathbb{E}_t [\Pi_{t,t+\tau}] &= \mathbb{E}_t \left[\int_t^{t+\tau} g_1(x_u) du \right] + \mathbb{E}_t \left[\int_t^{t+\tau} g_2(x_u) du \right] \\ &= \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \mathcal{L}^n [g_1(x_t)] + \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \mathcal{L}^n [g_2(x_t)]. \end{aligned} \tag{A.4}$$

Note that $g_1(x_t) = \alpha_1(V_t, \tau; y)S_t$, and $g_2(x_t) = \alpha_2(\eta_t, \tau; y)S_t$. By Lemma 1 of Bakshi and Kapadia (2003), $\mathcal{L}^n[g_1(x_t)]$ and $\mathcal{L}^n[g_2(x_t)]$ will also be proportional to S_t , which implies that:

$$\begin{aligned}\mathcal{L}^n[g_1(x_t)] &= \lambda^V V_t \Phi_{t,n}^V S_t \quad \forall n \\ \mathcal{L}^n[g_2(x_t)] &= \lambda^\eta \eta_t \Phi_{t,n}^\eta S_t \quad \forall n.\end{aligned}$$

Therefore, we have:

$$\begin{aligned}\mathbb{E}_t[\Pi_{t,t+\tau}] &= \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \mathcal{L}^n[g_1(x_t)] + \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \mathcal{L}^n[g_2(x_t)] \\ &= S_t [\lambda^V \beta_t^V V_t + \lambda^\eta \beta_t^\eta \eta_t],\end{aligned}$$

which implies that:

$$\frac{\mathbb{E}_t[\Pi_{t,t+\tau}]}{S_t} = \lambda^V \beta_t^V V_t + \lambda^\eta \beta_t^\eta \eta_t, \quad (\text{A.5})$$

where the sensitivities to the risk factors are given by:

$$\begin{aligned}\beta_t^V &= \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^V > 0 \\ \beta_t^\eta &= \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^\eta > 0.\end{aligned} \quad (\text{A.6})$$

The betas are positive since $\frac{\partial C_t}{\partial V_t} > 0$ and $\frac{\partial C_t}{\partial \eta_t} > 0$.

B Delta-Hedged VIX Options

The state vector is $x_t = [F_t \quad \eta_t]'$, and $g(x_t) = \frac{\partial C_t^*}{\partial \eta_t} \lambda^\eta$. Again, we will apply Itô-Taylor expansions. Let operators \mathcal{L} and Γ be such that:

$$\begin{aligned}\mathcal{L}[\cdot] dt &= \frac{\partial[\cdot]}{\partial F} \mu_{F,t} F_t dt + \frac{\partial[\cdot]}{\partial \eta} \gamma(\eta_t) dt + \frac{\partial[\cdot]}{\partial t} dt \\ &\quad + \frac{1}{2} \frac{\partial^2[\cdot]}{\partial F^2} [dF_t, dF_t] + \frac{1}{2} \frac{\partial^2[\cdot]}{\partial \eta^2} [d\eta_t, d\eta_t] + \frac{\partial^2[\cdot]}{\partial F \partial \eta} [dF_t, d\eta_t] \\ \Gamma[\cdot] &= \left[\frac{\partial[\cdot]}{\partial F} \sigma_{F_t}, \frac{\partial[\cdot]}{\partial \eta} \phi \sqrt{\eta_t} \right].\end{aligned} \quad (\text{B.1})$$

By Itô's Lemma, we have:

$$g(x_u) = g(x_t) + \int_t^u \mathcal{L}g(x_{u'}) du' + \int_t^u \Gamma g(x_{u'}) dW_{u'}.$$

Then, we have that:

$$\begin{aligned}
\int_t^{t+\tau} g(x_u) du &= \int_t^{t+\tau} \left(g(x_t) + \int_t^u \mathcal{L}g(x_{u'}) du' + \int_t^u \Gamma g(x_{u'}) dW_{u'} \right) du \\
&= \int_t^{t+\tau} g(x_t) du + \int_t^{t+\tau} \int_t^u \mathcal{L}g(x_{u'}) du' du + \int_t^{t+\tau} \int_t^u \Gamma g(x_{u'}) dW_{u'} du \\
&= g(x_t)\tau + \frac{1}{2}\mathcal{L}g(x_t)\tau^2 + \frac{1}{6}\mathcal{L}^\epsilon g(x_t)\tau^3 + \dots + \text{It\^o integrals} \\
&= \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \mathcal{L}^n g(x_t) + \text{It\^o integrals}.
\end{aligned} \tag{B.2}$$

This implies that $\mathbb{E}_t [\Pi_{t,t+\tau}] = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \mathcal{L}^n g(x_t)$, since the expectation of It\^o integrals is zero.

$$\mathbb{E}_t [\Pi_{t,t+\tau}^*] = \mathbb{E}_t \left[\int_t^{t+\tau} \frac{\partial C_s^*}{\partial \eta_s} \lambda_s^\eta ds \right].$$

Re-write Equation (2.18) using:

$$\begin{aligned}
g(x_t) &= \lambda_t^\eta \frac{\partial C_t^*}{\partial \eta_t} \\
&= \lambda^\eta \eta_t h_t^2(\tau; y) F_t,
\end{aligned}$$

where the second line follows from the homogeneity of $\frac{\partial C_t^*}{\partial \eta}$ in F_t , so $g(x_t) = \alpha(\eta_t, \tau; y) F_t$. By Lemma 1 of Bakshi and Kapadia (2003), $\mathcal{L}^n[g(x_t)]$ is also proportional to F_t , and we have that:

$$\begin{aligned}
\mathbb{E}_t [\Pi_{t,t+\tau}] &= \mathbb{E}_t \left[\int_t^{t+\tau} \frac{\partial C_s}{\partial \eta_s} \lambda_s^\eta ds \right] \\
&= \mathbb{E}_t \left[\int_t^{t+\tau} g(x_s) ds \right] \\
&= \lambda^\eta \eta_t F_t \beta_t^*,
\end{aligned} \tag{B.3}$$

where $\beta_t^* = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{n,t}$ which is positive since $\frac{\partial C_t^*}{\partial \eta} > 0$. This gives us the familiar factor model structure:

$$\frac{\mathbb{E}_t [\Pi_{t,t+\tau}]}{F_t} = \lambda^\eta \beta_t^* \eta_t. \tag{B.4}$$

C Risk-Neutral Skewness

The prices of the volatility, cubic, and quartic contracts $V_{t,t+\tau}, W_{t,t+\tau}, X_{t,t+\tau}$ are given

$$\begin{aligned}
V_{t,t+\tau} &= \int_{S_t}^{\infty} \frac{2(1 - \log \frac{K}{S_t})}{K^2} C(t, t + \tau; K) dK + \int_0^{S_t} \frac{2(1 + \log \frac{S_t}{K})}{K^2} P(t, t + \tau; K) dK, \\
W_{t,t+\tau} &= \int_{S_t}^{\infty} \frac{6 \log \frac{K}{S_t} - 3(\log \frac{K}{S_t})^2}{K^2} C(t, t + \tau; K) dK - \int_0^{S_t} \frac{6 \log \frac{S_t}{K} + 3(\log \frac{S_t}{K})^2}{K^2} P(t, t + \tau; K) dK, \\
X_{t,t+\tau} &= \int_{S_t}^{\infty} \frac{12(\log \frac{K}{S_t})^2 - 4(\log \frac{K}{S_t})^3}{K^2} C(t, t + \tau; K) dK + \int_0^{S_t} \frac{12(\log \frac{S_t}{K})^2 + 4(\log \frac{S_t}{K})^3}{K^2} P(t, t + \tau; K) dK,
\end{aligned}$$

and $\mu_{t,t+\tau} = e^{rf\tau} - 1 - \frac{e^{rf\tau}}{2}V_{t,t+\tau} - \frac{e^{rf\tau}}{6}W_{t,t+\tau} - \frac{e^{rf\tau}}{24}X_{t,t+\tau}$.

To construct these measures, we use out-of-the-money options to mitigate liquidity concerns. Following Shimko (1993), each day we interpolate the Black-Scholes implied volatility curve at the observable strikes using a cubic spline, and then calculate option prices to compute the above moments. We construct these measures for both S&P500 options and VIX options. Our implied volatility slope and risk-neutral skewness measures are calculated using options with the same maturity as our test assets.

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Tables and Figures

Table 1: **Summary Statistics**

| Variable | Mean | Std. Dev. | AR(1) | Corr. VIX^2 | Corr. $VVIX^2$ |
|------------|-------|-----------|-------|---------------|----------------|
| VIX^2 | 0.059 | 0.060 | 0.805 | 1.000 | 0.301 |
| $VVIX^2$ | 0.763 | 0.197 | 0.423 | 0.301 | 1.000 |
| RV^{SPX} | 0.031 | 0.060 | 0.620 | 0.880 | 0.316 |
| RV^{VIX} | 0.539 | 0.434 | 0.192 | 0.378 | 0.526 |

Table 1 gives summary statistics for the variance and jump measures we use. RV^{SPX} is the realized variance of S&P500 returns, calculated using 5-minute log futures returns. Monthly variables from 2006m2 to 2013m7. RV^{VIX} is the realized variance of VIX innovations, calculated using 5-minute log VIX index innovations. Realized measures are annualized, and overnight returns are excluded. AR(1) is the persistence of the variable. Corr. to VIX^2 is the correlation of the variable with VIX^2 , and Corr. to $VVIX^2$ is similarly defined. The model-free implied variance variable VIX^2 is defined as $(\frac{VIX}{100})^2$ and $VVIX^2$ is defined as $(\frac{VVIX}{100})^2$.

Table 2: **Predictability of Realized Measures**

| | VIX^2 | | $VVIX^2$ | | R_{adj}^2 |
|----------------------------------|---------|---------|----------|---------|-------------|
| | Slope | T-stat. | Slope | T-stat. | |
| Panel A: S&P500 Index | | | | | |
| $RV_{t,t+1}^{SPX}$ | 0.611 | [4.97] | | | 37.46 |
| | | | 0.066 | [1.57] | 3.60 |
| | 0.601 | [4.62] | 0.010 | [0.55] | 36.83 |
| Panel B: VIX Index | | | | | |
| $RV_{t,t+1}^{VIX}$ | 0.610 | [0.95] | | | -0.43 |
| | | | 0.799 | [4.23] | 12.20 |
| | -0.192 | [-0.30] | 0.817 | [3.52] | 11.23 |

Table 2 gives the realized measure predictability regressions. Monthly frequency sample spanning 2006m2 to 2013m6. The t-statistics shown are calculated using Newey and West (1987) robust standard errors with 6 lags. Realized measures calculated using high-frequency 5-minute data. Numbers in brackets are t-statistics.

Table 3: Delta-Hedged Option Gains

| | Moneyiness ($\frac{Strike}{Forward}$) | $\frac{\Pi}{S}$ (%) | t-stat. | Std. Dev. | AR(1) | $\frac{\Pi}{C}$ (%) | t-stat. |
|----------------------------------|---|---------------------|---------|-----------|-------|---------------------|----------|
| Panel A: S&P500 Index | | | | | | | |
| Call Options | 0.950 to 0.975 | 0.08 | [2.20] | 0.71 | 0.41 | 1.75 | [2.37] |
| | 0.975 to 1.000 | 0.01 | [0.50] | 0.60 | 0.26 | 0.76 | [0.82] |
| | 1.000 to 1.025 | -0.08 | [-3.32] | 0.54 | 0.11 | -5.87 | [-3.44] |
| | 1.025 to 1.050 | -0.13 | [-5.89] | 0.48 | 0.13 | -32.43 | [-7.18] |
| Put Options | 0.950 to 0.975 | -0.16 | [-4.84] | 0.72 | 0.38 | -18.27 | [-5.41] |
| | 0.975 to 1.000 | -0.21 | [-7.39] | 0.62 | 0.24 | -11.47 | [-6.48] |
| | 1.000 to 1.025 | -0.27 | [-9.79] | 0.59 | 0.05 | -9.81 | [-10.68] |
| | 1.025 to 1.050 | -0.30 | [-9.85] | 0.54 | 0.07 | -6.74 | [-10.23] |
| Panel B: VIX Index | | | | | | | |
| Call Options | 0.800 to 0.900 | -0.57 | [-3.07] | 2.07 | -0.09 | -3.44 | [-3.23] |
| | 0.900 to 1.000 | -1.35 | [-5.75] | 2.55 | -0.21 | -11.88 | [-5.73] |
| | 1.000 to 1.100 | -1.08 | [-3.72] | 2.86 | -0.09 | -12.56 | [-3.27] |
| | 1.100 to 1.200 | -1.41 | [-5.11] | 2.74 | -0.09 | -22.07 | [-4.48] |
| Put Options | 0.800 to 0.900 | -0.57 | [-1.79] | 2.49 | -0.06 | -13.32 | [-1.43] |
| | 0.900 to 1.000 | -1.28 | [-5.14] | 2.67 | -0.19 | -17.71 | [-4.27] |
| | 1.000 to 1.100 | -1.04 | [-3.39] | 2.99 | -0.12 | -7.50 | [-3.04] |
| | 1.100 to 1.200 | -1.30 | [-4.61] | 2.74 | -0.11 | -6.19 | [-4.40] |

Table 3 gives the delta-hedged option scaled gains. Monthly frequency sample spanning 2006m3 to 2012m10 for S&P500 options and VIX options. The t-statistics test the null hypothesis that the delta-hedged option scaled gain is equal to zero. Options have one month to maturity, and are held expiration to expiration. Π is the delta-hedged option gain, as given in Bakshi and Kapadia (2003). Delta values calculated using the Black-Scholes formula. On intermediate dates when an option price cannot be found, we fit a cubic polynomial to observed market implied volatilities to back out the option's implied volatility, which we use to calculate a delta for hedging purposes. The portfolio gains can be interpreted as an equal-weighted portfolio of all options whose moneyness falls inside the corresponding bin. The delta-hedge is rebalanced daily, with the margin difference earning the risk-free rate. $\frac{\Pi}{S}$ is the delta-hedged option gain scaled by the index, and $\frac{\Pi}{C}$ is the delta-hedged option gain scaled by the option price (for both puts and calls). Numbers in brackets are t-statistics that test the null hypothesis that the average delta-hedged gain is equal to zero.

Table 4: **Delta-Hedged Option Gains by Volatility Risk Sensitivities**

| | Vega = $\frac{\partial C}{\partial \sigma}$ | | Volga = $\frac{\partial^2 C}{\partial \sigma^2}$ | |
|-----------------------------|---|---------|--|---------|
| | slope | t-stat. | slope | t-stat. |
| Panel A: SPX Options | | | | |
| $\frac{\Pi_{t,t+1}}{S_t}$ | -0.051 | [-2.24] | | |
| | | | -0.007 | [-3.29] |
| | -0.178 | [-8.09] | -0.019 | [-8.17] |
| Panel B: VIX Options | | | | |
| $\frac{\Pi_{t,t+1}}{S_t}$ | -1.68 | [-4.12] | | |

Table 4 gives the delta-hedged S&P500 and VIX option gains cross-sectional regressions. Cross-sectional regression at monthly frequency sample spanning 2006m3 to 2012m10, with time fixed effects, using robust standard errors. Constant is omitted because zero risk sensitivity implies zero expected delta-hedged option gain. The dependent variable is delta-hedged option gain $\Pi_{t,t+\tau}$, which is calculated as described in the data section, and scaled gains are given in percentages. Panel A is for S&P500 options. The independent variables are vega and volga, as calculated from Black-Scholes. Panel B is for VIX options. The independent variable in Panel B is vega (of VIX) calculated from Black-Scholes. Numbers in brackets are t-statistics.

Table 5: Predictability of Delta-Hedged SPX Option Gains

| | VIX^2 | | $VVIX^2$ | | $GAINS_{t-1}$ | |
|--|---------|---------|----------|---------|---------------|---------|
| | slope | t-stat. | slope | t-stat. | slope | t-stat. |
| Panel A: SPX Options | | | | | | |
| $GAINS_t = \frac{\Pi_{t,t+\tau}}{S_t}$ | -0.78 | [-2.83] | | | 0.29 | [4.17] |
| | | | -0.51 | [-4.64] | 0.31 | [5.18] |
| | -0.47 | [-2.30] | -0.49 | [-4.85] | 0.34 | [4.79] |
| Panel B: VIX Options | | | | | | |
| $GAINS_t = \frac{\Pi_{t,t+\tau}}{S_t}$ | | | -1.37 | [-3.57] | -0.07 | [-3.40] |

Table 5 gives the delta-hedged S&P500 option gains (Panel A) and VIX option gains (Panel B) predictability regressions. Panel regression at monthly frequency sample spanning 2006m3 to 2012m10, with cross-sectional fixed effects, using robust standard errors. The dependent variable is delta-hedged option gain $\Pi_{t,t+\tau}$, which is calculated as described in the data section, and scaled gains are given in percentages. The cross-sectional identifier is the moneyness bin of the option, as given in Table 3. Gains for each bin are averaged, which can be interpreted as the gain on an equally-weighted portfolio of options within a given moneyness bin. The independent variables are $VIX^2 = \left(\frac{VIX}{100}\right)^2$, $VVIX^2 = \left(\frac{VVIX}{100}\right)^2$ which are in annualized percentage squared units. Following Bakshi and Kapadia (2003), lagged gains are included to correct for serial correlation of the residuals. Numbers in brackets are t-statistics.

Table 6: Predictability of Realized Measures - Alternate Specifications

| | <i>VIX</i> | | <i>VVIX</i> | | $\ln VIX$ | | $\ln VVIX$ | |
|----------------------------------|------------|---------|-------------|---------|-----------|---------|------------|---------|
| | slope | t-stat. | slope | t-stat. | slope | t-stat. | slope | t-stat. |
| Panel A: S&P500 Index | | | | | | | | |
| $\sigma_{t,t+1}^{SPX}$ | 0.736 | [6.50] | | | | | | |
| | | | 0.179 | [1.51] | | | | |
| | 0.732 | [6.04] | 0.011 | [0.19] | | | | |
| $\ln \sigma_{t,t+1}^{SPX}$ | | | | | 0.961 | [8.72] | | |
| | | | | | | | 0.514 | [1.01] |
| | | | | | 0.969 | [8.07] | -0.110 | [-0.40] |
| Panel B: VIX Index | | | | | | | | |
| $\sigma_{t,t+1}^{VIX}$ | 0.210 | [0.83] | | | | | | |
| | | | 0.748 | [4.23] | | | | |
| | -0.014 | [-0.06] | 0.751 | [3.50] | | | | |
| $\ln \sigma_{t,t+1}^{VIX}$ | | | | | 0.047 | [0.55] | | |
| | | | | | | | 0.735 | [3.62] |
| | | | | | -0.006 | [-0.08] | 0.739 | [3.14] |

Table 6 gives the realized measure predictability regressions for alternative volatility specifications (volatility and log-volatility). Monthly frequency sample spanning 2006m2 to 2013m6. The t-statistics shown are calculated using Newey and West (1987) robust standard errors with 6 lags. Realized measures calculated using high-frequency 5-minute data. VIX and VVIX are in quoted index units divided by 100, and can be interpreted as percentage annual volatility. Numbers in brackets are t-statistics.

Table 7: **Robustness to Jump Measures**

| | VIX^2 | $VVIX^2$ | $SLOPE$ | $SKEW$ | RJ | $GAINS_{t-1}$ |
|--|---------------|---------------|---------------|---------------|---------------|---------------|
| | slope t-stat. | slope t-stat. | slope t-stat. | slope t-stat. | slope t-stat. | slope t-stat. |
| Panel A: SPX Options | | | | | | |
| $GAINS_t = \frac{\Pi_{t,t+\tau}}{S_t}$ | -0.63 [-2.12] | -0.42 [-7.09] | -2.25 [-1.54] | | | 0.33 [4.91] |
| | -0.51 [-2.13] | -0.48 [-5.10] | | 0.01 [0.96] | | 0.34 [4.70] |
| | -0.47 [-2.28] | -0.48 [-5.30] | | | 0.01 [0.71] | 0.34 [4.74] |
| Panel B: VIX Options | | | | | | |
| $GAINS_t = \frac{\Pi_{t,t+\tau}}{S_t}$ | | -1.36 [-3.48] | -0.69 [-0.36] | | | -0.07 [-3.27] |
| | | -1.37 [-3.55] | | -0.08 [-0.88] | | -0.08 [-3.09] |
| | | -1.37 [-3.56] | | | -0.01 [-0.51] | -0.07 [-3.40] |

Table 7 gives the delta-hedged S&P500 option gains (Panel A) and VIX option gains (Panel B) predictability regressions. Panel regression at monthly frequency sample spanning 2006m3 to 2012m10, with cross-sectional fixed effects, using robust standard errors. The dependent variable is delta-hedged option gain $\Pi_{t,t+\tau}$, which is calculated as described in the data section, and scaled gains are given in percentages. The cross-sectional identifier is the moneyness bin of the option, as given in Table 3. Gains for each bin are averaged, which can be interpreted as the gain on an equally-weighted portfolio of options within a given moneyness bin. The independent variables are $VIX^2 = \left(\frac{VIX}{100}\right)^2$, $VVIX^2 = \left(\frac{VVIX}{100}\right)^2$ which are in annualized percentage squared units. For S&P500 options, $SLOPE$ is calculated as the Black and Scholes (1973) implied volatility of an out-of-the-money ($\frac{K}{S} = 0.9$) minus the implied volatility of an at-the-money put option ($\frac{K}{S} = 1$). For VIX options, $SLOPE$ is calculated using the difference in implied volatility between a $\frac{K}{S} = 1.1$ call option and an at-the-money call option. RJ is calculated using high-frequency data as in Barndorff-Nielsen and Shephard (2004) and Wright and Zhou (2009); the calculation for S&P500 options uses S&P500 futures tick data, while the calculation for VIX options uses VIX tick data. $SKEW$ is calculated using the model-free method of Bakshi et al. (2003). Following Bakshi and Kapadia (2003), lagged gains are included to correct for serial correlation of the residuals. Numbers in brackets are t-statistics.

Figure 1: Time Series Plot

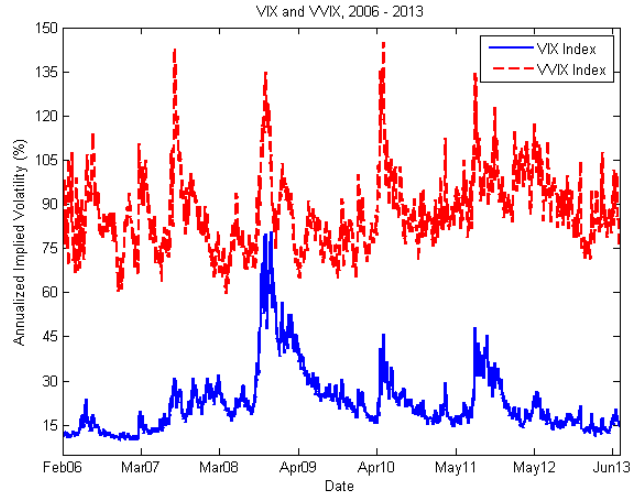


Figure 1 plots VIX and VVIX index from 2006m2 to 2013m6. The solid blue line is VIX and the dashed red line is VVIX. VVIX data from 2007m1 onwards is from CBOE, and for 2006 it is calculated from VIX options using the method published by the CBOE. All VIX data is from official CBOE VIX levels.

Figure 2: Realized Measures

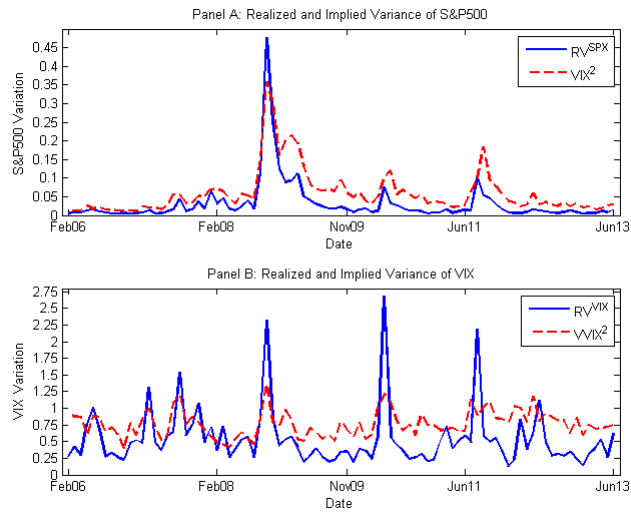


Figure 5 plots realized and implied variances for the S&P500 (top panel) and VIX (bottom panel). Monthly data from 2006m2 to 2013m6. The blue solid lines are realized variances, and red dashed lines are the model-free implied variances. Realized variances calculated from 5-minute high-frequency data. All measures in annualized variance units.

Figure 3: Vega and Gamma by moneyness

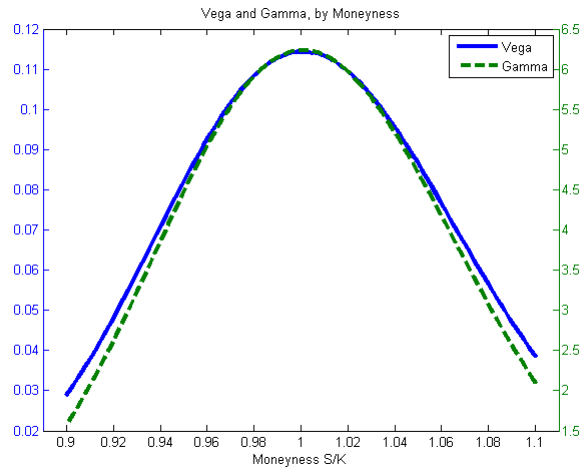


Figure 2 plots Black-Scholes vega and gamma by option moneyness for average levels of volatility. Although levels are different, the shape (hence cross-sectional dispersion) of the variables are nearly identical.

Figure 4: Volga by moneyness

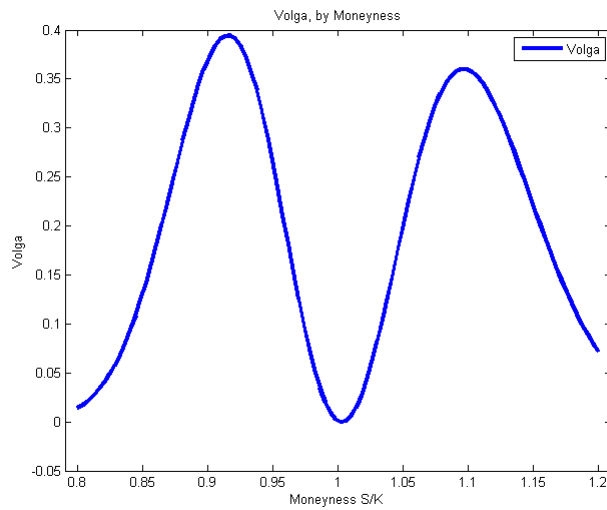


Figure 3 Black-Scholes volga (volatility gamma) by option moneyness for average levels of volatility.